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THE ENTRY OF AN ELLIPTICAL PARABOLOID INTO A LIQUID AT VARIABLE VELOCITY†

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Within framework of the Wagner approximation, an exact solution is constructed for the problem of an elliptical paraboloid entering an ideal, incompressible liquid at variable velocity. It is shown that the region of contact of the liquid with the body is elliptical for any law of penetration. The loads experienced by a freely falling three-dimensional body upon impact with the liquid surface are determined. It is shown that the maximum acceleration of the penetrating body is governed by a single parameter which depends on the conditions of impact and the shape of the body. © 2002 Elsevier Science Ltd. All rights reserved.

Experimental research [1, 2] has shown that the forces exerted on a penetrating blunt body by a liquid reach their maximum values at fairly small depths of penetration.

The liquid flow at the initial stage of entry is determined using an approach developed by Wagner [3], which amounts to reducing the complex initial fluid dynamics problem to the problem of a floating disk, whose shape and size vary with time, impacting the surface of a liquid. The unknown shape of the disk is determined at each instant of time from the condition of continuous contiguity of the free and rigid parts of the liquid boundary (Wagner's condition). It can be shown that Wagner's condition is equivalent to stipulating that the displacements of the liquid particles must be finite [4]. Formally speaking, the equations and boundary conditions of Wagner's theory are obtained by linearizing the equations and boundary conditions of the initial non-linear problem, written in terms of the velocity potential, on the assumption that the ratio of the depth of penetration to the horizontal dimension of the contact region is small. The limits of applicability of Wagner's theory were analysed in detail in [2, 5].

Another approximate entry theory was proposed by Karman [6]. Karman's theory ignores the variation of the wetted part of the penetrating body due to the opposing lift of the free surface of the liquid. On that assumption, the contact region coincides with the section cut from the moving body by the plane $z' = 0$ and is assumed to be known.

Wagner's approach was highly recommended for solving practical problems [2]. However, in the case of an unknown dividing line, such as boundary conditions, Wagner's problem is still quite difficult. It is non-linear, irrespective of whether the equation of motion and boundary conditions are linearized.

The special feature of the entry problem is that the body-liquid contact region is unknown and has to be determined together with the flow of the liquid and the form of its free surface. Until recently, all studies, even in the context of Wagner's approach, were confined to the case of two-dimensional or axisymmetric bodies. The case of an elliptical paraboloid is fairly general, since the shape of an arbitrary blunt body may be approximately replaced by an elliptical paraboloid near the point of initial contact with the liquid. This approximation is justified at small depths of penetration.

The problem of an elliptical paraboloid entering water at a constant velocity was considered, within the framework of Wagner's approximation, in [7]. It was assumed there that the contact region is an ellipse at each instant of time. The solution constructed satisfies the condition at the free surface of the liquid only approximately. It has been shown [8] that Wagner's problem has self-similar solutions at entry velocities that vary with time according to a power law, for bodies, the shape of whose surface is described by positive and homogeneous functions. It is claimed [8] that the solution of Wagner's problem for an elliptical paraboloid is self-similar, but the contact region is not elliptical. This claim was made because the inverse problem method proposed in [8] for investigating entry processes led to cumbersome formulae, which were not subjected to a detailed analysis. The inverse problem method, in which it is required to determine the shape of a body from the given entry velocity and the shape of

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the contact region, has received further development [9]. The case of an elliptical contact region was considered in a fair degree of detail. It was shown [9] that, for an elliptical paraboloid entering a liquid half-space at constant velocity or constant acceleration, the contact region is elliptical. The accuracy of the approximate results obtained in [7] was investigated in [9]; it was shown, in particular, that the drag force computed by the approximate formulae is practically the same as its exact value [9] for all elongations of the penetrating elliptical paraboloid.

It will be shown below that the method proposed in [7] enables one to construct an exact solution of Wagner's problem for an elliptical paraboloid entering a liquid half-space at arbitrary velocity. The solution will be obtained on the assumption that the contact region is an ellipse at each instant of time. The question of whether the solution of Wagner's problem is unique will not be considered. The solution constructed will be used to estimate the maximum loads experienced by a convex body of finite dimensions falling freely and subsequently penetrating the surface of a liquid.

1. FORMULATION OF THE PROBLEM

We will consider the three-dimensional problem of an elliptical paraboloid impacting the free boundary of an ideal incompressible liquid. At the initial instant of time ($t' = 0$) the liquid occupies the half-space $z' < 0$ and is at rest; the rigid body is tangent to the free surface of the liquid $z' = 0$ at a single point, which is taken to be the origin of a Cartesian system of coordinate $Ox'y'z'$. The initial velocity of the body, V_0, is given. The body is moving vertically downward under the action of gravity and the drag force exerted by the liquid. The mass M of the body and its principal radii of curvature R_x and R_y are assumed to be given. The weight and surface tension of the liquid will be ignored. It is required to determine the flow of the liquid and the drag force at the initial stage of penetration.

The problem is considered in dimensionless variables (distinguished from the dimensional variables by the omission of the prime). The coordinate axes Ox and Oy are oriented in such a way that the radii of curvature R_x and R_y of the elliptical paraboloid in the sections $y = 0$ and $x = 0$, respectively, satisfy the inequality $R_y \geqslant R_x$. As the length scale of R_x and the time scale we take the quotient R_x/V_0, where $V_0 = U'(0)$ is the velocity of the body at the time of impact. In dimensionless variables, the position of the body is described by the equation

$$z = \tfrac{1}{2}x^2 + \tfrac{1}{2}(1-\varepsilon^2)y^2 - h(t) \tag{1.1}$$

where $\varepsilon = \sqrt{1 - R_x/R_y}$ is the eccentricity of the horizontal sections of the elliptical paraboloid and $h(t)$ is the dimensionless depth of penetration, $h' = R_x h$. The dimensional entry velocity $U'(t')$ is equal to $V_0 v(t)$, where $v(t) = dh/dt$ and $t = t'V_0/R_x$.

Within the framework of the Wagner approximation the liquid flow due to the body's entry is described by a velocity potential $\varphi(x, y, z, t)$ which satisfies the relations

$$\Delta\varphi = 0 \qquad (z < 0) \tag{1.2}$$

$$\varphi = 0 \qquad (z = 0, \quad (x, y) \notin D(t)) \tag{1.3}$$

$$\varphi_z = -v(t) \qquad (z = 0, \quad (x, y) \in D(t)) \tag{1.4}$$

$$\sqrt{x^2 + y^2 + z^2}\,\varphi \to 0 \qquad (x^2 + y^2 + z^2 \to \infty) \tag{1.5}$$

where $D(t)$ is the contact region, within which the linearized impermeability condition (1.4) is satisfied. The part of the plane $z = 0$ outside the contact region corresponds to the free liquid boundary, where the pressure remains constant throughout the whole motion. The hydrodynamic pressure is defined in Wagner's theory by the linearized Cauchy–Lagrange integral $p(x, y, z, t) = -\varphi_t(x, y, z, t)$. Accordingly, condition (1.3) is a linearized dynamic condition on the free boundary of the liquid. The shape of the free boundary, $z = \eta\,(x, y, t)$, is determined from the linearized kinematic condition

$$\eta_t = \varphi_z\,(x, y, 0, t) \qquad (z = 0, (x, y) \notin D(t) \tag{1.6}$$

The continuous contiguity condition for the free and rigid parts of the liquid boundary yields an equation for the shape of the contact region

$$\tfrac{1}{2}x^2 + \tfrac{1}{2}(1-\varepsilon^2)y^2 - h(t) = \eta(x,y,t) \quad ((x,y)\in\Gamma(t)) \tag{1.7}$$

where $\Gamma(t)$ is the boundary of the contact region (the contact curve).

The solution of Wagner's problem (1.2)–(1.7) will be sought in the class of functions describing flows with finite kinetic energy. This means that the velocity potential $\varphi(x, y, z, t)$ is a continuous function up to the boundary $z = 0$, and the velocity of the liquid particles $\nabla\varphi$ has at most a root singularity on the contact curve $\Gamma(t)$. In accordance with the basic hypotheses [2, 5] that lead to Wagner's problem, one can expect the solution of problem (1.2)–(1.7) to provide a correct description of the initial stage of the process, when the depth of penetration h(t) is much less than the size of the region $D(t)$.

Even after linearization of the equations of motion and boundary conditions, the immersion problem is still non-linear and quite complicated. One of the difficulties in investigating the problem stems from the singularity of the derivatives of the solution on a moving boundary curve such as the boundary condition on the curve $\Gamma(t)$. In order to regularize Wagner's problem, we will define the displacement potential as

$$\phi(x,y,z,t) = \int_0^t \varphi(x,y,z,\tau)d\tau \tag{1.8}$$

The gradient of the displacement potential yields the vector of displacements of liquid particles $X = \nabla\phi$, $X = (X, Y, Z)$. The displacement potential is widely used in the linear dynamic theory of elasticity (see, e.g., [10, 11]). It is clear that the smoothness of the displacement potential as a function of the time variable t is greater by 1 than that of $\varphi(x, y, z, t)$. It is less obvious that integration with respect to time in (1.8) increases smoothness with respect to the space variables x, y, z as well. The transformation (1.8) is a smoothing one, since the singularities of the velocity potential are concentrated on the curve $\Gamma(t)$, which is moving in the plane $z = 0$.

The boundary-value problem for the displacement potential

$$\Delta\phi = 0 \qquad (z<0) \tag{1.9}$$

$$\phi = 0 \qquad (z=0, \quad (x,y)\notin D(t)) \tag{1.10}$$

$$\phi_z = \tfrac{1}{2}x^2 + \tfrac{1}{2}(1-\varepsilon^2)y^2 - h(t) \qquad (z=0, \quad (x,y)\in D(t)) \tag{1.11}$$

$$\sqrt{x^2+y^2+z^2}\,\phi \to 0 \qquad (x^2+y^2+z^2 \to \infty) \tag{1.12}$$

is obtained from Eqs (1.2)–(1.5) by integrating them with respect to time taking into account kinematic condition (1.6) and Wagner's condition (1.7). Details of this procedure were described in [12]. Note that the time t occurs in problem (1.9)–(1.12) as a parameter, that is, the displacement potential, unlike the velocity potential, may be defined at every instant of time, irrespective of the previous history of the process. Problem (1.9)–(1.12) was obtained [7] using the Lagrangian approach to the description of fluid flow. The problem has the same form as (1.2)–(1.5), but now the impermeability condition (1.11) explicitly depends on the shape of the immersed body, while Wagner's condition (1.7) becomes a restriction on the class of unknown functions

$$\phi \in C^2(R^3_-)\cup C^1(\overline{R}^3_-) \tag{1.13}$$

where $R^3_- = \{x, y, z \mid z < 0\}$ and $\overline{R}^3_- = \{x, y, z \mid z \leqslant 0\}$. Condition (1.13) means that it is required to determine a solution of the problem with mixed boundary conditions (1.9)–(1.12), describing the flow of the liquid on the assumption that the field of displacements of liquid particles is continuous up to the boundary.

In problem (1.9)–(1.13) it is sufficient to determine the boundary $\Gamma(t)$ of the contact region $D(t)$ and the value of the displacement potential $\phi(x, y, 0, t)$ in that region. Boundary-value problem (1.9)–(1.12) may be reduced to a singular integral equation of the first kind in the function $\overline{\phi}_{xx} + \overline{\phi}_{yy}$, where $\overline{\phi}(x, y, t) = \phi(x, y, 0, t)$. After solving this integral equation the function $\overline{\phi}$ is defined as the solution of Poisson's equation satisfying the following boundary conditions

$$\overline{\phi} = 0, \quad \partial\overline{\phi}/\partial n = 0 \quad ((x,y)\in\Gamma(t)) \tag{1.14}$$

which follow from (1.13) and condition (1.10) at the free surface. Here $\partial\overline{\phi}/\partial n$ is the normal derivative of the unknown function on the plane curve $\Gamma(t)$. It is obvious that one of conditions (1.14) serves to construct the function $\bar{\phi}(x, y, t)$ for the given region $D(t)$, while the other is used to determine the geometry of that region, which guarantees the required smoothness of the solution.

After a solution of problem (1.9)–(1.13) has been constructed, the velocity potential $\varphi(x, y, z, t)$ and hydrodynamic pressure $p(x, y, z, t)$ are constructed using the formulae

$$\varphi = \phi_t, \quad p = -\phi_{tt} \quad (z \leqslant 0) \tag{1.15}$$

The velocity $v(t)$ of the body and its depth $h(t)$ of penetration are determined using Newton's Second Law, which yields, in dimensionless variables,

$$v_t = \beta - \gamma F(t), \quad h_t = v(t) \quad (t > 0); \quad v(0) = 1, \quad h(0) = 0 \tag{1.16}$$

where

$$\beta = \frac{gR_x}{V_0^2}, \quad \gamma = \frac{\rho_0 R_x^3}{M}, \quad F(t) = \int\int_{D(t)} p(x, y, 0, t) dx dy \tag{1.17}$$

(ρ_0 is the density of the liquid, M is the mass of the body and $F(t)$ is the dimensionless drag force exerted on the immersed body by the liquid). Solving problem (1.16) taking into account relations (1.13), (1.15) and (1.17), we obtain

$$v(t) = 1 + \beta t + \gamma \int\int_{D(t)} \varphi(x, y, 0, t) dx dy \tag{1.18}$$

$$h(t) = t + \tfrac{1}{2}\beta t^2 + \gamma \int\int_{D(t)} \phi(x, y, 0, t) dx dy \tag{1.19}$$

Equation (1.19) must be taken together with Wagner's problem (1.9)–(1.13) and considered simultaneously with it.

Problem (1.9)–(1.13), (1.19) is quite complicated, even for the simplest three-dimensional shape of the penetrating body. A solution of the problem will be sought on the assumption that the contact region $D(t)$ is elliptical. The question of the uniqueness of this solution remains open.

2. AN ELLIPTICAL CONTACT REGION

Let the boundary $\Gamma(t)$ of the contact region be describe by the equation

$$x^2/a^2(t) + y^2/b^2(t) = 1 \tag{2.1}$$

where the semi-axes $a(t)$ and $b(t)$ of the ellipse have to be determined. It is assumed that the eccentricity $e = \sqrt{1 - a^2(t)/b^2(t)}$ of the ellipse (2.1) is independent of time. The solution of problem (1.9)–(1.13), (1.19), (2.1) will be sought in the form

$$\phi = b^3(t)\Phi(\lambda, \mu, \nu), \quad b(t) = b_0\sqrt{h(t)} \tag{2.2}$$

$$\lambda = x / b(t), \quad \mu = y / b(t), \quad \nu = z / b(t), \quad a(t) = \sqrt{1 - e^2}\, b(t)$$

The constants e and b_0 have to be determined during the course of solving the problem.

The boundary-value problem for the new unknown function $\Phi(\lambda, \mu, \nu)$ is

$$\Delta\Phi = 0 \qquad (\nu < 0) \tag{2.3}$$

$$\Phi = 0 \quad (\nu = 0, \ (\lambda, \mu) \notin D_0, \quad D_0 = \{\lambda, \mu \mid (1 - e^2)^{-1}\lambda^2 + \mu^2 < 1\}) \tag{2.4}$$

$$\Phi_\nu = \tfrac{1}{2}(\lambda^2 + (1 - \varepsilon^2)\mu^2) - b_0^{-2} \quad (\nu = 0, \ (\lambda, \mu) \in D_0) \tag{2.5}$$

$$\sqrt{\lambda^2 + \mu^2 + \nu^2}\,\Phi \to 0 \quad (\lambda^2 + \mu^2 + \nu^2 \to \infty) \tag{2.6}$$

$$\Phi \in C^2(R_-^3) \cup C^1(\bar{R}_-^3) \tag{2.7}$$

Problem (2.3)–(2.7) contains a single parameter ε, which depends on the shape of the body, and two unknown quantities, b_0 and e, which must be considered as functions of ε, where $0 \leqslant \varepsilon < 1$. It is important to observe that the solution of problem (2.3)–(2.7) depends not on the law of motion of the body but only on its shape. A solution of problem (2.3)–(2.6), which approximately satisfies condition (2.4) at the free boundary and exactly satisfies relations (2.3), (2.5) and (2.6), was constructed in [7], but it certainly does not satisfy condition (2.7). In what follows we will construct an exact solution of problem (2.3)–(2.7).

Consider the vertical component of the vector of displacements, $Z(\lambda, \mu, \nu) = \Phi_\nu(\lambda, \mu, \nu)$. It is a harmonic function in the lower half-plane, given in the contact region D_0, and its normal derivative $Z_\nu(\lambda, \mu, 0)$ vanishes on the free boundary of the liquid, $\nu = 0$, $(\lambda, \mu) \notin D_0$, which is corollary of Eq. (2.3) and condition (2.4). In addition, $(\lambda^2 + \mu^2 + \nu^2)Z(\lambda, \mu, \nu) \to 0$ at infinity, as follows from (2.6). Continuing Z as an even function into the upper half-space, we arrive at a Dirichlet problem outside the elliptical disk D_0. A solution of the latter has been constructed [7] using Lamé functions, as follows (the symbol $\Sigma_\pm$ denotes summation of the terms with the upper and lower sign):

$$Z(\lambda, \mu, \nu) = \sum_\pm L_\pm \left(\frac{\lambda^2}{\alpha_\pm - e^2} + \frac{\mu^2}{\alpha_\pm} + \frac{\nu^2}{\alpha_\pm - 1} - 1 \right) \frac{\sigma_\pm(\rho)}{\sigma_\pm(1)} \tag{2.8}$$

with the condition

$$b_0 = \sqrt{6/(2 - e^2 - \varepsilon^2)} \tag{2.9}$$

Here

$$L_\pm = \frac{\alpha_\pm(\alpha_\pm - e^2)(e^2 - \alpha_\mp \varepsilon^2)}{2e^2(\alpha_\pm - \alpha_\mp)}, \quad \alpha_\pm = \frac{1}{3}(1 + e^2 \pm \sqrt{1 - e^2 + e^4}) \tag{2.10}$$

$$\sigma_\pm(\rho) = \int_\rho^\infty \frac{d\rho}{(\rho^2 - \alpha_\pm)^2 \sqrt{(\rho^2 - e^2)(\rho^2 - 1)}}$$

where $\rho = \rho(\lambda, \mu, \nu)$ is the root with maximum absolute value of the cubic equation in ρ^2

$$\frac{\lambda^2}{\rho^2 - e^2} + \frac{\mu^2}{\rho^2} + \frac{\nu^2}{\rho^2 - 1} = 1 \tag{2.11}$$

In this situation $\rho = 1$ corresponds to the contact region D_0 and $\rho > 1$ on the free boundary. Solution (2.8) is continuous up to the boundary of the elliptical disk D_0. The solution is not unique, since the parameter e remains undefined.

In order to find the boundary value $\bar{\Phi}(\lambda, \mu) = \Phi(\lambda, \mu, 0)$ of the unknown function, we rewrite Eq. (2.1) in the form

$$\Phi_{\lambda\lambda} + \Phi_{\mu\mu} = -Z_\nu(\lambda, \mu, \nu) \quad (\nu < 0) \tag{2.12}$$

and consider its limit as $\nu \to 0$ separately in the contact region, $(\lambda, \mu) \in D_0$, and on the free boundary. The right-hand side of (2.12) vanishes on the free boundary, but its limit as $\nu \to 0$ in the contact region may be calculated using formulae (2.8)–(2.11). We have

$$\lim_{\nu \to 0} Z_\nu(\lambda, \mu, \nu) = \sum_\pm L_\pm \left(\frac{\lambda^2}{\alpha_\pm - e^2} + \frac{\mu^2}{\alpha_\pm} - 1 \right) \frac{1}{\sigma_\pm(1)} \lim_{\nu \to 0} \left(\frac{d\sigma_\pm}{d\rho} \frac{\partial\rho}{\partial\nu} \right) \quad ((\lambda, \mu) \in D_0)$$

where the derivative $\partial\rho/\partial\nu$ is evaluated using Eq. (2.11). We obtain

$$\lim_{\nu \to 0} \left(\frac{d\sigma_\pm}{d\rho} \frac{\partial\rho}{\partial\nu} \right) = \frac{1}{g(\lambda, \mu, e)(1 - \alpha_\pm)^2 \sqrt{1 - e^2}}, \quad g(\lambda, \mu, e) = \left(1 - \frac{\lambda^2}{1 - e^2} - \mu^2 \right)^{1/2}$$

which enables us to write the equation for $\Phi(\lambda, \mu)$ in the form

$$\overline{\Phi}_{\lambda\lambda} + \overline{\Phi}_{\mu\mu} = f(\lambda, \mu) \quad ((\lambda, \mu) \in D_0) \tag{2.13}$$

where

$$\begin{gathered} f(\lambda, \mu) = -\frac{C_1\lambda^2 + C_2\mu^2 - C_3}{g(\lambda, \mu, e)}, \quad C_1 = a_0[\alpha_+ a_+ - \alpha_- a_-] \\ C_2 = a_0[(\alpha_+ - e^2)a_+ - (\alpha_- - e^2)a_-], \quad C_3 = a_0[\alpha_+(\alpha_+ - e^2)a_+ - \alpha_-(\alpha_- - e^2)a_-] \\ a_0^{-1} = 2e^2(\alpha_+ - a_-)\sqrt{1 - e^2}, \quad a_\pm = \frac{e^2 - \alpha_\mp \varepsilon^2}{\sigma_\pm(1)(1 - \alpha_\pm)^2} \end{gathered} \tag{2.14}$$

The function $\Phi(\lambda, \mu)$ will satisfy condition (2.4) on the free boundary and condition (2.7), that is, it will be continuously differentiable for all values of λ and μ, if and only if

$$\overline{\Phi} = 0, \quad \partial\overline{\Phi}/\partial n = 0 \quad ((\lambda, \mu) \in \Gamma_0), \tag{2.15}$$

where Γ_0 is the boundary of D_0 and $\partial\Phi/\partial n$ is the normal derivative of the unknown function on Γ_0.

Boundary-value problem (2.13)–(2.15) for the function $\Phi(\lambda, \mu)$ in the contact region is over-determined, and its solution exists only provided a certain additional condition is satisfied. To deduce this condition, we multiply both sides of Eq. (2.13) by an arbitrary harmonic function $\omega(\lambda, \mu)$ and integrate the result over D_0, using Green's formula and boundary conditions (2.15). This gives

$$\iint_{D_0} f(\lambda, \mu)\omega(\lambda, \mu)d\lambda d\mu = 0 \tag{2.16}$$

Hence, a necessary condition for problem (2.13)–(2.15) to be solvable is that the function $f(\lambda, \mu)$ must be orthogonal to any harmonic function.

Let us consider two linearly independent harmonic functions $\omega_1(\lambda, \mu) = 1$ and $\omega_2(\lambda, \mu) = \lambda^2 - \mu^2$. Substituting these functions into integral (2.16) and evaluating the integrals taking formulae (2.14) into account, we find that

$$\begin{gathered} C_1(1 - e^2) + C_2 = 3C_3 \\ (1 - e^2)(2 - 3e^2)C_1 - (2 + e^2)C_2 + 5e^2C_3 = 0 \end{gathered} \tag{2.17}$$

It was be verified that the first equality of (2.17) is satisfied identically by virtue of the second, third and fourth relations in (2.14), while the second equality of (2.17) leads to the equation

$$\begin{gathered} \varepsilon^2 = e^2 \frac{M_+(e)/M_-(e) - 1}{\alpha_+ M_+(e)/M_-(e) - \alpha_-} \\ M_\pm(e) = [2(2 - e^2)\alpha_\mp + e^2 - 3]\sigma_\pm(1)(1 - \alpha_\pm)^2 \end{gathered} \tag{2.18}$$

which may be used to determine the eccentricity e of the contact region as a function of the eccentricity ε of the elliptical paraboloid.

The graph of the function $e = e(\varepsilon)$ is represented in Fig. 1 by the continuous curve. The dashed curve represents the approximation $\varepsilon = e/\sqrt{\alpha_+(e)}$ obtained in [7].

Taking Eqs (2.17) into consideration, we can write the first formula of (2.14) in the form

$$f(\lambda, \mu) = \left(\frac{\partial^2}{\partial\lambda^2} + \frac{\partial^2}{\partial\mu^2}\right)(-N(e)g^3(\lambda, \mu, e)), \quad N(e) = \frac{C_3(1 - e^2)}{3(2 - e^2)} \tag{2.19}$$

which clearly shows that Eq. (2.18) guarantees that condition (2.16) is satisfied with an arbitrary harmonic function $\omega(\lambda, \mu)$.

Equation (2.13), in which $f(\lambda, \mu)$ substituted from formula (2.19), and boundary conditions (2.15) give

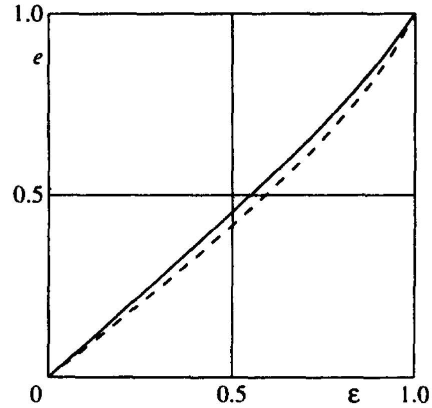

Fig. 1

$$\overline{\Phi}(\lambda, \mu) = -N(e)g^3(\lambda, \mu, e) \tag{2.20}$$

which completes the construction of the solution of problem (2.3)–(2.7).

Formulae (2.2) and (2.20) enable us to express the potential of displacements in the contact region in the form

$$\phi(x, y, 0, t) = -N(e)b^3(t)\left[1 - \frac{x^2}{a^2(t)} - \frac{y^2}{b^2(t)}\right]^{3/2}$$

The velocity potential in the contact region is determined using relations (1.15), (1.16) and (2.2)

$$\varphi(x, y, 0, t) = -\frac{9N(e)}{(2 - e^2 - \varepsilon^2)\sqrt{1 - \varepsilon^2}} a(t)\, v(t)\left[1 - \frac{x^2}{a^2(t)} - \frac{y^2}{b^2(t)}\right]^{1/2} \tag{2.21}$$

This formula may be obtained by considering the solution of problem (1.2)–(1.5) for the elliptical contact region (2.1) as the limit of the solution of the classical problem of an ellipsoid moving in an unbounded liquid [13] when one of the axes of the ellipsoid tends to zero. The limiting solution is identical with (2.21) when

$$N(e) = \tfrac{1}{9}\sqrt{1 - e^2}\,(2 - e^2 - \varepsilon^2)\,/\,\mathbf{E}(e)$$

where $\mathbf{E}(e)$ is an elliptic integral of the first kind. Computations have confirmed this equality, but a rigorous proof is difficult because of the complicated form of the function $N(e)$. This equality will be used below to express the results in a more compact form.

In particular, using formula (2.21), we find the added mass of the expanding elliptical disk (2.1) in dimensional variables to be

$$M_a(t) = (2\pi / 3)\rho_0[a'(t)]^2 b'(t) / \mathbf{E}(e)$$

that is, it equals half the mass of the liquid in an ellipsoid with semi-axes a', b' and $a'/\mathbf{E}(e)$, where $a' = R_x a$ and $b' = R_x b$. Among all elliptical disks with the same area ($\pi a'b' = S$), the greatest added mass is possessed by a circular disk ($e = 0$). The added mass of the elliptical disk

$$M_a = \rho_0 S^{3/2} m(e), \quad m(e) = 2(1 - e^2)^{1/4} / (3\sqrt{\pi}\mathbf{E}(e))$$

decreases monotonically as the eccentricity e increases (Fig. 2). The value $m(0) = 4/(3\pi^{3/2})$ corresponds to a circular disk. It can be seen that $m(e)$ differs significantly from $m(0)$ only for very elongated elliptical disks. Computations show that the ratio $(m(0) - m(e))/m(0)$ is positive and does not exceed 0.001 for $e < 0.368$ ($\varepsilon < 0.4$), 0.01 for $e < 0.6$ ($\varepsilon < 0.66$) or 0.1 for $e < 0.885$ ($\varepsilon < 0.92$). Consequently, the added mass of an elliptical disk may be calculated by using the following approximate rule: the added mass

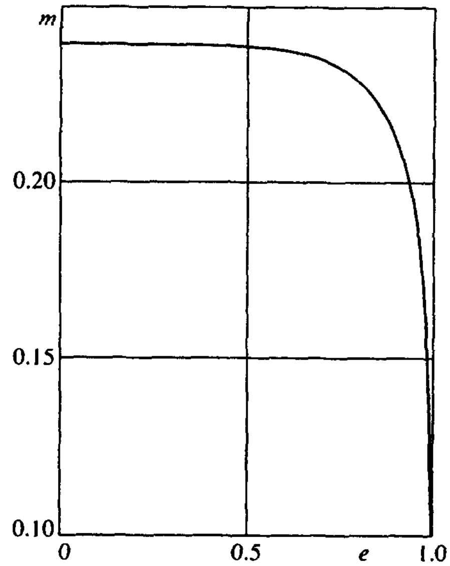

Fig. 2

of an elliptical disk differs only slightly from that of a circular disk of the same area. We may expect the overloads experienced by an axisymmetric body on impact with water to be reduced by a flattening of the body in one of the horizontal directions, without a change in its volume.

Indeed, let us consider a paraboloid of revolution $z' = (x'^2 + y'^2)/(2R)$ and an elliptical paraboloid $z' = (x'^2(2R_x) + y'^2(2R_y)$, where $R_x = R\sqrt{1-\varepsilon^2}$, $R_y = R/\sqrt{1-\varepsilon^2}$, $0 \leqslant \varepsilon < 1$, the areas of whose cross-sections are identical. When these paraboloids penetrate into a liquid at constant velocity V_0, they experience hydrodynamic forces $F'(t', 0)$ and $F'(t', \varepsilon)$, respectively. Using the third formula of (1.17) we find that

$$F'(t', \varepsilon) = \chi(\varepsilon)F'(t', 0)$$

$$\chi(\varepsilon) = \frac{\pi\sqrt{2}}{\mathbf{E}(e)}(1-\varepsilon^2)^{1/4}\frac{1-e^2}{1-\varepsilon^2}\left[1+\frac{1-e^2}{1-\varepsilon^2}\right]^{-3/2}$$

Computations show that $\chi(0) = 1$ and the function $\chi(\varepsilon)$ decreases monotonically as ε increases. Consequently, the drag force decreases when the cross-section of a penetrating body of revolution experiences an elliptical deformation. A similar property holds for the drag of equivalent bodies in supersonic flow [14]. It should be noted that a marked decrease in the drag force is observed only for significant deformations, when ε is close to unity: by 5% when $\varepsilon = 0.83$ and 10% when $\varepsilon = 0.9$.

Within the limits of the Karman approximation [2] (ignoring the opposing lift of the free boundary), the liquid flow is described by the solution of problem (1.2)–(1.5), where the contact region coincides with the section of the penetrating body (1.1) by the plane $z = 0$, that is

$$D_K(t) = \left\{x, y \left| \frac{1}{2}x^2 + \frac{1}{2}(1-\varepsilon^2)y^2 < h(t)\right.\right\}$$

The subscript K indicates quantities related to the Karman approximation.

The distribution of the velocity potential over the region $D_K(t)$ is given by the formula

$$\varphi_K(x, y, 0, t) = -\frac{v(t)}{\mathbf{E}(\varepsilon)}[2h(t) - x^2 - (1-\varepsilon^2)y^2]^{1/2}$$

which enables one to determine the drag force exerted on the penetrating body by the liquid and to set up an equation of motion for the body. We would expect Wagner's approach to yield an upper bound for the loads experienced by the body on penetration and Karman's approach to yield a lower bound.

3. OVERLOADS EXPERIENCED BY A BODY ENTERING WATER

Substituting expressions (2.2) and (2.20) into Eq. (1.19), we obtain

$$h(t) = t + \frac{1}{2}\beta t^2 - \gamma q(\varepsilon) h^{5/2}(t) \tag{3.1}$$

$$q(\varepsilon) = \frac{8\sqrt{6}\pi}{5} \frac{1 - e^2(\varepsilon)}{\mathrm{E}(e)[2 - e^2 - \varepsilon^2]^{3/2}} \tag{3.2}$$

The function $q(\varepsilon)$ depends only on the shape of the body and increases without limit as $\varepsilon \to 1$. A graph of the function $k(\varepsilon) = q(0)/q(\varepsilon)$ is shown in Fig. 3.

The form of the equation of motion (3.1) shows that within the framework of Wagner's approximation the law of penetration into water for an elliptical paraboloid will be the same as for a sphere of radius R_x with mass $k(\varepsilon)M$, where $k(\varepsilon) < 1$.

Equation (3.1) can be represented in a simpler form if we introduce extended variables $s = (\gamma q)^{2/3} h$, $\tau = (\gamma q)^{2/3} t$ and the parameter $\alpha = \beta(\gamma q)^{2/3}/2$. We obtain a quadratic equation

$$\alpha\tau^2 + \tau = s^{5/2} + s \tag{3.3}$$

in $\tau = \tau(s)$, where $\tau(0) = 0$. In the new variables, the form of the function $\tau(s)$ depends on the single parameter α, which characterizes the influence of gravity on the body's motion. The dimensional acceleration w' of the elliptical paraboloid on entering water is given by the formula

$$\frac{w'}{g} = -\frac{\tau_{ss}(s)}{2\alpha\tau_s^3(s)} \tag{3.4}$$

The right-hand side of this expression depends on the parameter α and on the depth of penetration s. Accordingly, the maximum acceleration of the penetrating body, $w'_{\max}$, depends only on α, which is convenient when one is comparing experimental data obtained under different conditions.

The continuous curve in Fig. 4 is a graph of the ratio $|w'_{\max}|/g$ as a function of α.

Note that

$$\frac{|w'_{\max}|}{g} \sim \frac{320}{243(20)^{1/3}}\alpha^{-1} \quad (\alpha \to 0) \tag{3.5}$$

The function on the right of asymptotic formula (3.5) is shown in Fig. 4 as a dashed curve. Formula (3.5) gives a relative error of less than 5% when $\alpha < 0.04$.

In the axisymmetric case, $\varepsilon = 0$ and $q(0) = 8\sqrt{3}/5$. Computations using formula (3.4) for experimental conditions with a freely falling sphere on an unperturbed liquid surface [1], corresponding to $\alpha = 0.0396$ in the notation adopted here, give $|w'_{\max}|/g \approx 11.65$. The experimental value of the same quantity is approximately 9.64.

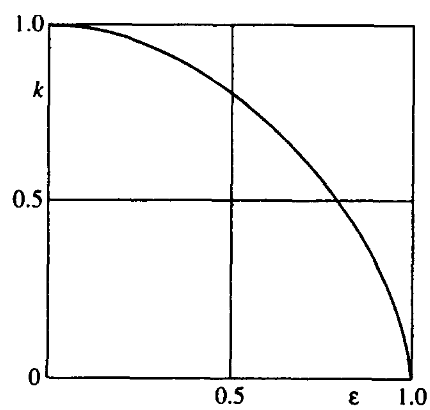

Fig. 3

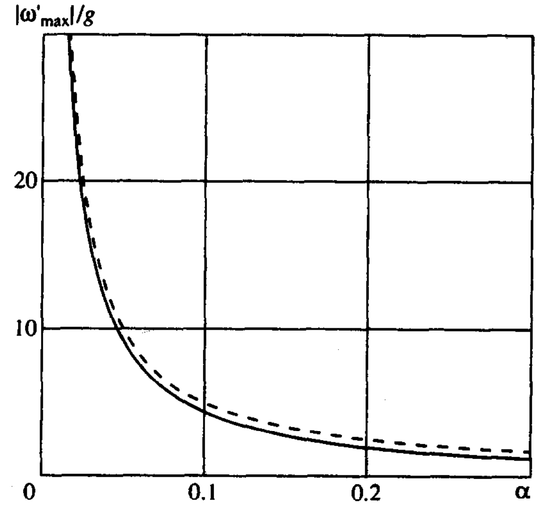

Fig. 4

The theoretical results obtained in [1, 2] are in better agreement with the experimental data, but they are applicable only to axisymmetric bodies. It was assumed in the calculations in [1] that the distribution of the velocity potential over a circular contact region is faithfully described by Wagner's theory. However, the complete Cauchy-Lagrange equation was used to evaluate the drag force, not the linearized equation as in this paper. This approach is also applicable to the water-entry problem for an elliptical paraboloid. When that is done the equation of motion of the body will be somewhat more complicated than (3.3). The simple formula (3.5) may be used in planning experiments for a preliminary estimate of the loads on three-dimensional bodies entering water.

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